Grafting hyperbolic metrics and Eisenstein series

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Abstract

The family hyperbolic metric for the plumbing variety $\{zw = t\}$ and the non holomorphic Eisenstein series $E(\zeta; 2)$ are combined to provide an explicit expansion for the hyperbolic metrics for degenerating families of Riemann surfaces. Applications include an asymptotic expansion for the Weil-Petersson metric and a local form of symplectic reduction.

1 Introduction

A genus g, d punctured Riemann surface with $2g - 2 + d$ positive admits a complete hyperbolic structure. A family of marked hyperbolic structures degenerates provided there are closed geodesics with lengths tending to zero. Degenerating families provide an important means for investigating global quantities and the global geometry of the moduli space of Riemann surfaces. Degenerating families also provide a means for investigating families of structures with boundary lengths tending to zero. Hyperbolic structures with geodesic boundary can be doubled across boundaries to form structures without boundary. The examinations are based on expansions for degenerating structures.

L. Bers considered certain degenerating families of marked hyperbolic structures [Ber74]. He found that away from collars about short geodesics the structures limit to hyperbolic structures with cusps with neighborhoods of cusps removed. Expansions have been presented for such degenerating families of hyperbolic structures. We combine the approaches of [Wol91, Wol90] to present an explicit expansion in terms of the hyperbolic metrics (see (2) and (3) below) of fibers of the plumbing family $\mathcal{V}_0$ and the

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non holomorphic Eisenstein series $E(\zeta; 2)$. Beginning with a Riemann surface R with a formal pairing of n pairs of punctures (additional unpaired punctures of R are permitted), we use the fiberwise gluing of $\mathcal{V}_0$ to $R - \{cusp\ neighborhoods\}$ to construct a degenerating family $\{R_t\}$. We then use geometric-interpolation to combine the $\mathcal{V}_0$ fiber metrics with the R metric to form grafted metrics dg_t^2. A counterpart interpolation the *melding* is introduced for the Eisenstein series. The construction differs from Thurston's grafting of projective structures for Riemann surfaces [Gol87]. We then use uniqueness properties for the prescribed curvature equation to show that the grafted metric and meldings combine to provide an expansion.

Theorem. *For a grafting cutoff function η and all t small the hyperbolic metric ds_{hyp}^2 of R_t has the expansion*

$$ds_{hyp}^2 = dg_t^2 \big(1 + \frac{4\pi^4}{3} \sum_{k=1}^n (\log |t_k|)^{-2}(E_{k,1}^{\dagger} + E_{k,2}^{\dagger}) + O(\sum_{k=1}^n (\log |t_k|)^{-3})\big).$$

The functions $E_{k,1}^{\dagger}$ and $E_{k,2}^{\dagger}$ are the meldings of the Eisenstein series $E(\cdot; 2)$ associated to the pair of cusps plumbed to form the kth collar. The O-term refers to the intrinsic C^1-norm of a function on R_t. The bound depends on the choice of η and a lower bound for the injectivity radius for the complement of the cusp regions in R.

A length expansion is a first application. For the plumbing family $\{R_t\}$ there is a comparison between fibers by domain inclusion of collar complements. The comparison provides a setting for relating lengths for geodesics disjoint from the collars

$$\ell_\alpha(\ell) = \ell_\alpha(0) + \sum_{k=1}^n \frac{\ell_k^2}{6} \int_\alpha (E_{k,1} + E_{k,2})ds + O(\sum_{k=1}^n \ell_k^3).$$

Similarly in [Wol94] Wolf's infinite energy harmonic map degeneration expansion was employed with Eisenstein series to study the variation of eigenvalues of the Laplacian.

A further application is for the Kähler and symplectic geometry of the Teichmüller space $\mathcal{T}$ of punctured Riemann surfaces with hyperbolic metrics [Ahl61, Bus92, IT92, Nag88]. The infinitesimal deformations of a surface R are represented by the Beltrami differentials $\mathcal{H}(R)$ harmonic with respect to the hyperbolic metric [Ahl61]. The cotangent space of $\mathcal{T}$ at R is $Q(R)$ the space of holomorphic quadratic differentials with at most simple poles

at the punctures of R. The holomorphic tangent-cotangent pairing is

$$(\mu, \varphi) = \int_R \mu\varphi$$

for $\mu \in \mathcal{H}(R)$ and $\varphi \in Q(R)$. Elements of $\mathcal{H}(R)$ are symmetric tensors given as $\overline{\varphi}(ds^2)^{-1}$. A Hermitian form for $\mathcal{H}(R)$ defines a Hermitian metric for Teichmüller space. For $\mu, \nu \in \mathcal{H}(R)$ and the hyperbolic area element dA the Weil-Petersson (WP) form is

$$\langle \mu, \nu \rangle_{WP} = \int_R \mu\bar{\nu}\, dA, \qquad [\text{Ahl61}]$$

and for R with punctures $p_1, \ldots, p_d$ and corresponding Eisenstein series $E_1, \ldots, E_d$ the Takhtajan-Zograf (TZ) form for the puncture p_j is

$$\langle \mu, \nu \rangle_{TZ,p_j} = \int_R \mu\bar{\nu}E_j\, dA, \qquad [\text{TZ88}].$$

The Takhtajan-Zograf metric is $\sum_{j=1}^{d} \langle\ ,\ \rangle_{TZ,p_j}$.

The WP metric is Kähler and non complete with non pinched negative curvature. The metric has been studied extensively; see [Wol03, Wol06] for beginning references. The WP metric provides a complete $CAT(0)$ geometry for the augmented Teichmüller space $\overline{\mathcal{T}}$ [DW03, Wol03, Yam04]. The mapping class group is the WP isometry group and the augmented Teichmüller space is the closed convex hull of the set of totally noded Riemann surfaces [MW02, Wol03]. Degenerating families of hyperbolic metrics have been used to develop an asymptotic expansion for the metric and its curvature [DW03, Huaar, Mas76, LSY04, LSY05, Sch93, Wol90, Wol03].

The TZ metric is Kähler and non complete [Obi99, Obi01, TZ88]. L. Takhtajan and P. Zograf introduced the metric in their investigation of the local index formula for $\overline{\partial}$ and the Quillen metric [TZ88, TZ91]. The authors showed for punctured Riemann surfaces that the Chern forms for the determinant line bundles for families of $\overline{\partial}$-operators are sums of the WP and TZ Kähler forms [TZ91]. K. Obitsu, W.-K. To and L. Weng have used degenerating families of hyperbolic metrics to develop an asymptotic expansion for the TZ metric [OTW06]. In the Deligne-Mumford moduli space of stable curves transverse to the divisor $\mathcal{D}$ of noded curves the TZ metric is smaller than the WP metric.

We apply the above hyperbolic metric expansion to refine the asymptotic expansion for the WP metric. Prior expansions [Mas76, OTW06, Wol03] focus on directions transverse to the divisor $\mathcal{D}$. We consider directions parallel

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to $\mathcal{D}$. For the family $\{R_t\}$ we express the expansion in terms of geodesic length with the expansion $\ell = 2\pi^2(-\log|t|)^{-1} + O((\log|t|)^{-2})$ [Wol90, Ex. 4.3].

Theorem. *In the Deligne-Mumford moduli space of stable curves for tangent subspaces parallel to the divisor of noded curves the WP metric has the expansion for all small ℓ*

$$\langle \ , \ \rangle_{WP}^{parallel}(\ell) = \langle \ , \ \rangle_{WP}(0) + \sum_{k=1}^{n} \frac{\ell_k^2}{3}(\langle \ , \ \rangle_{TZ,k,1} + \langle \ , \ \rangle_{TZ,k,2}) + O(\sum_{k=1}^{n} \ell_k^3)$$

for the TZ forms for punctures paired to form the kth node.

There are considerations for WP geometry. The augmentation locus of $\overline{\mathcal{T}}$ is a union of closed convex strata [Wol03, Yam04]. A complete convex subset of a $CAT(0)$ space is the base of a projection [BH99, Chap. II.2]. A stratum in $\overline{\mathcal{T}}$ is described as the 0-level set for a collection σ of geodesic lengths $\{\ell_k\}_{k \in \sigma}$. Distance to the stratum is given as $(2\pi \sum_{k \in \sigma} \ell_k)^{1/2} + O(\sum_{k \in \sigma} \ell_k^{5/2})$ [Wol03, Coro. 21] and [Wolom, Coro. 4.10] with the projection distance non increasing. Accordingly the perturbation in $\sum_{k \in \sigma} \ell_k$ of the WP metric from the 0-level set is expected to be non negative. From the above parallel to the level set the initial perturbation is strictly positive.

There are considerations for Kähler forms based on a second characterization of TZ forms. For a holomorphic family of punctured Riemann surfaces the family of cotangent spaces along a puncture p defines a tautological holomorphic line bundle ψ_p over the base of the family. The hyperbolic metric can be renormalized to define a metric $\| \ \|_{ren,p}$ for the cotangent space at the puncture. The Chern form for the family of cotangent spaces along the puncture p is given in terms of the TZ exterior 2-form $c_1(\| \ \|_{ren,p}) = \frac{4}{3}\omega_{TZ,p}$ [Wol07]. The asymptotic expansion for Kähler forms becomes

$$\omega(\ , \ )_{WP}^{parallel}(\ell) = \omega(\ , \ )_{WP}(0) + \sum_{k=1}^{n} \frac{\ell_k^2}{4}\big(c_1(\psi_{k,1}) + c_1(\psi_{k,2})\big) + O(\sum_{k=1}^{n} \ell_k^3). \quad (1)$$

An application is for the volume of moduli spaces of bordered Riemann surfaces. M. Mirzakhani considered the moduli space $\mathcal{M}(b_1, \ldots, b_d)$ of bordered Riemann surfaces with geodesic boundary components of prescribed lengths $(b_1, \ldots, b_d)$ [Mir07a]. The spaces $\mathcal{M}(b_1, \ldots, b_d)$ form an $(\mathbb{R}_{\geq 0})^d$ bundle over $\mathcal{M}$ the moduli space of genus g, d punctured Riemann surfaces. A Riemann surface with geodesic boundary and a *point* on each boundary is alternately described by a d punctured Riemann surface and a product of

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d factors of S^1, a principal torus-bundle over a punctured Riemann surface. In [Mir07b, Secs. 3 & 4] Mirzakhani used symplectic reduction (for an $(S^1)^d$ quasi-free action following Guillemin-Sternberg) to provide a simple description for the family of WP Kähler forms

$$\omega_{WP}\big|_{\mathcal{M}(b_1,\ldots,b_d)} = \omega_{WP}\big|_{\mathcal{M}} + \sum_j \frac{b_j^2}{4} c_1(\psi_j).$$

The expansion is a relation for cohomology classes on $\overline{\mathcal{M}}$ and is presented for a general choice of principal connection for the S^1 bundles. Formula (1) is a local form of the symplectic reduction.

2 The plumbing family hyperbolic metric

For a Riemann surface R with a finite area hyperbolic metric and punctures there is a natural *cusp coordinate* (with germ unique modulo rotation) at each puncture: at a puncture p, the coordinate z with $z(p) = 0$ and the hyperbolic metric of R given as $(\frac{|dz|}{|z|\log|z|})^2$. For the surface uniformized by the upper half-plane $\mathbb{H}$ with p represented by a width-one cusp at infinity then a loop once encircling the puncture corresponds to the deck transformation $\zeta \to \zeta + 1$; the natural cusp coordinate is given as $z = e^{2\pi i\zeta}$ for ζ the natural $\mathbb{H}$ coordinate.

We recall the construction of the plumbing family [Wol90, esp. Sec. 2.4] and the grafting of hyperbolic metrics [Wol90, esp. Sec. 3.3]. Considerations begin with the *plumbing variety* $\mathcal{V} = \{(z, w, t) \mid zw = t, \ |z|, |w|, |t| < 1\}$. The defining function $zw - t$ has differential $z\,dw + w\,dz - dt$. Consequences are that $\mathcal{V}$ is a smooth variety, (z, w) are global coordinates, while (z, t) and (w, t) are not. Consider the projection $\Pi : \mathcal{V} \to D$ onto the t-unit disc. The projection Π is a submersion, except at $(z, w) = (0, 0)$; we consider $\Pi : \mathcal{V} \to D$ as a (degenerate) family of open Riemann surfaces. The t-fiber, $t \neq 0$, is the hyperbola germ $zw = t$ or equivalently the annulus $\{|t| < |z| < 1, w = t/z\} = \{|t| < |w| < 1, z = t/w\}$. The 0-fiber is the intersection of the unit ball with the union of the coordinate axes in $\mathbb{C}^2$; on removing the origin the union becomes $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$. Each fiber of $\mathcal{V}_0 = \mathcal{V} - \{0\} \to D$ has a complete hyperbolic metric:

for $t \neq 0$, on $\{|t| < |z| < 1\}$ then

$$ds_t^2 = \left(\frac{\pi}{\log |t|} \csc \frac{\pi \log |z|}{\log |t|} \left| \frac{dz}{z} \right| \right)^2 ;$$

for $t = 0$, on $\{0 < |z| < 1\} \cup \{0 < |w| < 1\}$ then

$$ds_0^2 = \left(\frac{|d\zeta|}{|\zeta| \log |\zeta|} \right)^2 \quad \text{for } \zeta = z, \, w.$$

(2)

The family of hyperbolic metrics (ds_t^2) is a continuous metric, degenerate only at the origin, for the vertical line bundle of $\mathcal{V}$. In particular we have the elementary expansion

$$ds_t^2 = \left(\frac{|d\zeta|}{|\zeta| \log |\zeta|} \right)^2 \left(\Theta \csc \Theta \right)^2 \quad \text{for} \quad \Theta = \frac{\pi \log |z|}{\log |t|}$$
$$= ds_0^2 \left(1 + \frac{1}{3} \Theta^2 + \frac{1}{15} \Theta^4 + \dots \right).$$

(3)

Consider R a finite union of Riemann surfaces with punctures. A plumbing family is the fiberwise gluing of the complement of cusp neighborhoods in R and the plumbing variety $\mathcal{V} = \{(z, w, t) \mid zw = t, |z|, |w|, |t| < 1\}$. For a positive constant $c_* < 1$ and initial surface R, with puncture p with cusp coordinate z and puncture q with cusp coordinate w, we construct a family $\{R_t\}$. For $|t| < c_*^4$ the resulting surface R_t will be independent of c_*; the constant c_* will serve to specify the overlap of coordinate charts and to define a *collar* in each R_t.

We first describe the gluing of fibers. For $|t| < c_*^4$, remove from R the punctured discs $\{0 < |z| \leq |t|/c_*\}$ about p and $\{0 < |w| \leq |t|/c_*\}$ about q to obtain a surface R_{t/c_*}^* [Wol90, Sec. 2.4, second parag.]. For $t \neq 0$, form an identification space R_t, by identifying the annulus $\{|t|/c_* < |z| < c_*\} \subset R_{t/c_*}^*$ with the annulus $\{|t|/c_* < |w| < c_*\} \subset R_{t/c_*}^*$ by the rule $zw = t$. The resulting surface R_t is the *plumbing* for the prescribed value of t. We note for $|t| < |t'|$ that there is an inclusion of R_{t/c_*}^* in R_{t'/c_*}^*; the inclusion maps provide a way to compare structures on the surfaces. The inclusion maps are a basic feature of the plumbing construction. We next describe the plumbing family. Consider the variety $\mathcal{V}_{c_*} = \{(z, w, t) \mid zw = t, |z|, |w| < c_*, |t| < c_*^4\}$ and the disc $D_{c_*} = \{|t| < c_*^4\}$. The complex manifolds $M = R_{t/c_*}^* \times D_{c_*}$ and $\mathcal{V}_{c_*}$ have holomorphic projections to the disc D_{c_*}. The variables z, w denote prescribed coordinates on R_{t/c_*}^* and on $\mathcal{V}_{c_*}$. There are holomorphic maps of

subsets of M to $\mathcal{V}_{c_*}$, commuting with the projections to D_{c_*}, as follows

$$(z,t) \xrightarrow{\hat{F}} (z, t/z, t) \text{ and } (w,t) \xrightarrow{\hat{G}} (w, t/w, t).$$

The identification space $\mathcal{F} = M \cup \mathcal{V}_{c_*}/\{\hat{F}, \hat{G} \text{ equivalence}\}$ is the *plumbing family* $\{R_t\}$ with projection to D_{c_*} (an analytic fiber space of Riemann surfaces in the sense of Kodaira [Kod86]). For $0 < |t| < c_*^4$, the t-fiber of $\mathcal{F}$ is the surface R_t constructed by overlapping annuli.

The hyperbolic metric ds^2 of R restricted to $R_{c_*}^*$ can be combined with the fiber hyperbolic metric ds_t^2 of $\mathcal{V}$ restricted to $\mathcal{V}_{c_*^2}$ to provide an approximation, the grafted metric, to the hyperbolic of R_t, [Wol90, Sec. 3.3]. To this purpose select $\eta(a)$, a smooth function, identically unity for $a \leq a_0 < 0$ and identically zero for $a \geq 0$. (We further restrict t to satisfy $e^{2a_0}c_*^2 \geq |t|$.) For the variable z denoting the cusp coordinate of R_{t/c_*}^* and the fiber coordinate for $\mathcal{V}_{c_*}$ define a smooth metric in a neighborhood of $\{|z| = c_*\}$ by the rule

$$dg_t^2 = (ds^2)^{1-\eta}(ds_t^2)^{\eta} \text{ with } \eta = \eta(\log(|z|/c_*));$$

and similarly in a neighborhood of $\{|w| = c_*\}$ (for a Riemann surface a metric is a section of an $\mathbb{R}^+$-bundle with structure group $\mathbb{R}^{\times}$; the geometric-interpolation is well defined). In the *collar bands* $\{e^{a_0}c_* \leq |z| \leq c_*\}$ and $\{e^{a_0}c_* \leq |w| \leq c_*\}$ the *grafted metric* dg_t^2 is the geometric-interpolation of the two metrics. Away from the collar the grafted metric is prescribed as ds^2, the R hyperbolic metric, and in the complement of the bands in the collar the grafted metric is prescribed as ds_t^2.

The grafted metric is the combination of the metric ds^2 for R_{t/c_*}^* and the fiber metric ds_t^2 for $\mathcal{V}_{c_*^2}$. The basic relation $ds_t^2 = ds_0^2(\Theta \csc \Theta)^2$ for $\Theta = \frac{\pi \log |z|}{\log |t|}$ for the fiber metrics of $\mathcal{V}$ provides that ds_t^2 is real analytic in $\epsilon = \frac{\pi}{\log |t|}$, locally for nonzero z. For ζ the coordinate for $\mathbb{H}$ and $\Theta = \frac{-2\pi^2 \Im\zeta}{\log |t|}$ the relation provides $ds_t^2 = ds_0^2(1 + \frac{4\pi^4}{3}(\log |t|)^{-2}(\Im\zeta)^2 + O((\log |t|)^{-4}))$ a presage of the expansion in the next section. The construction of the grafted metric is motivated by two important properties: for t small, $dg_t^2 = ds^2(1 + O((\log |t|)^{-2}))$, and in the collar core dg_t^2 coincides with the fiber metrics for $\mathcal{V}$. Importantly the curvature of the grafted metric is identically -1 on the complement of the collar bands, and satisfies in the collar bands

$$K_{\text{graft}} = -1 - \frac{\epsilon^2}{6}\Lambda + O(\epsilon^4) \tag{4}$$

for $\Lambda = \frac{\partial}{\partial a}(a^4 \frac{\partial}{\partial a}\eta)$, for $a = \log |z|, \log |w|$, [Wol90, Sec. 3.4.MG, Def'n 3.8].

The construction of the grafted metric is local for a pair of punctures. The considerations apply for plumbing multiple pairs of punctures.

3 The special truncation of $E(\zeta; 2)$

The Eisenstein series $E(\zeta; s)$ for $s = 2$ plays a central role in our considerations. We begin with standard estimates. For a Fuchsian group Γ, with the stabilizer of infinity the group of integer-translations $\Gamma_{\mathbb{Z}}$, consider the relative Poincaré series

$$E(\zeta; 2) = \sum_{A \in \Gamma_{\mathbb{Z}} \backslash \Gamma} (\Im A(\zeta))^2, [\text{Bor97, Ven90}].$$

The function $(\Im \zeta)^2$ on $\mathbb{H}$ is an eigenfunction of the hyperbolic Laplacian with eigenvalue 2. The above series is associated to the cusp at infinity. In general eigenfunctions φ on $\mathbb{H}$ with eigenvalue λ satisfy a mean-value property; consequently for ϵ_0 positive an eigenfunction φ satisfies

$$|\varphi(\zeta_0)| \leq c(\epsilon_0, \lambda) \int_{B(\zeta_0; \epsilon_0)} |\varphi| d\mathcal{A}$$

for B the metric ball and $d\mathcal{A}$ the hyperbolic area element [Fay77]. A consequence is that the Eisenstein series converges locally uniformly on $\mathbb{H}$. Uniform bounds for the Eisenstein series are also a further consequence. To this purpose we consider the injectivity radius of the projection to $\mathbb{H}/\Gamma$ of a point ζ. The cusp region $\{\Im \zeta > 1\}$ is *precisely invariant* by Γ: the translate of the region by an element of Γ is either disjoint from the region, or coincides with the region and the transformation is an integer-translation. Accordingly for $\Im \zeta > 1$ the injectivity radius of the image point on $\mathbb{H}/\Gamma$ is approximately $(2\Im \zeta)^{-1}$. Accordingly a point of $\mathbb{H}/\Gamma$ with injectivity radius at least ϵ_0 has lifts with Γ-orbit contained in $\{2\Im \zeta \leq \epsilon_0^{-1}\}$; the bound is uniform among all Fuchsian groups with stabilizer of infinity $\Gamma_{\mathbb{Z}}$. Accordingly there is a universal bound at points with injectivity radius at least ϵ_0

$$E(\zeta_0; 2) \leq c(\epsilon_0) \int_{B(\zeta_0; \epsilon_0/2)} E d\mathcal{A}$$
$$\leq c(\epsilon_0) \int_{\cup_{A \in \Gamma_{\mathbb{Z}} \backslash \Gamma} A(B(\zeta_0; \epsilon_0/2))} (\Im \zeta)^2 d\mathcal{A} \leq c(\epsilon_0) \int_{2\Im \zeta \leq e^{\epsilon_0/2} \epsilon_0^{-1}; 0 \leq \Re \zeta \leq 1} d\mathcal{E}$$

for $d\mathcal{E}$ the Euclidean area element. We are interested in bounds for all cusp regions of $\mathbb{H}/\Gamma$ uniform in Γ. By successive conjugations of Γ, the cusps of Γ

can be represented at infinity with stabilizer $\Gamma_{\mathbb{Z}}$. For a cusp so represented at infinity, the quotient space $\{\Im\zeta > 1\}/\Gamma_{\mathbb{Z}}$ embeds in $\mathbb{H}/\Gamma$ to determine a *cusp region*. Cusp regions for distinct cusps are disjoint.

The Eisenstein series, transformed for a cusp represented at infinity, has the expansion

$$E(\zeta;2) = \delta_\infty (\Im\zeta)^2 + \hat{e}(\zeta)$$

with δ_∞ the indicator for the defining cusp for the series, and $\hat{e}(\zeta)$ bounded as $O((\Im\zeta)^{-1})$ for $\Im\zeta$ large [Bor97, Ven90]. We write $E^\star$ for the discontinuous function that satisfies $E^\star(\zeta;2) = E(\zeta;2) - (\Im\zeta)^2$ for the cusp region defining the series represented at infinity and that otherwise agrees with E.

Lemma 1. *For ζ the $\mathbb{H}$ coordinate and a cusp represented by a width-one cusp at infinity the derivatives of order at most k of $E^\star$ for $\Im\zeta > 2$ are bounded as $O((\Im\zeta)^{-1})$. The bound depends on k but is independent of Γ.*

Proof. The supremum bound is the main consideration. For the metric ball of radius $\frac{1}{2}$, the mean-value estimate provides that $E^\star$ is suitably bounded on $\Im\zeta = \frac{3}{2}$. Accordingly for a suitable positive constant c we have that $-c(\Im\zeta)^{-1} \le E^\star \le c(\Im\zeta)^{-1}$ on $\Im\zeta = \frac{3}{2}$ with $E^\star$ and $(\Im\zeta)^{-1}$ each vanishing at infinity. The pair of functions $E^\star$ and $(\Im\zeta)^{-1}$ are eigenfunctions of the hyperbolic Laplacian with the positive eigenvalue 2. By the maximum principle $E^\star - c(\Im\zeta)^{-1}$, resp. $E^\star + c(\Im\zeta)^{-1}$, realizes its maximum, resp. minimum, on the boundary of the domain $\{\Im\zeta \ge \frac{3}{2}\}$, [GT01]. Realized at infinity the maximum , resp. minimum, is zero. The supremum bound $-c(\Im\zeta)^{-1} \le E^\star \le c(\Im\zeta)^{-1}$ for $\Im\zeta \ge \frac{3}{2}$ is a consequence. The bound for the derivatives of $E^\star$ will be a consequence of the supremum-bound interior estimates [BJS64, Chap. 5] and the eigenequation. In particular for $B \subset \mathbb{R}^2$, a bounded domain, and $B_0 \subset\subset B$ there exists a constant $c = c(B, B_0)$ such that for Δ the Euclidean Laplacian then $\|P\|_{1,B_0} \le c(\|P\|_{0,B} + \|\Delta P\|_{0,B})$, where $\|\ \|_k$ is the Euclidean C^k-norm for a domain, [BJS64, Chap. 5]. To make the desired analysis, Euclidean translate $E^\star$ to a fixed ball. The C^1-bound immediately follows from the eigenequation. More generally, for $\Delta_m = \frac{\partial^m}{\partial\zeta^p \partial\bar{\zeta}^{m-p}}$, a derivative of order m, we have that $\Delta\Delta_m E^\star = \Delta_m \Delta E^\star = \Delta_m 2(\Im\zeta)^{-2} E^\star$. It follows that $\Delta\Delta_m E^\star$ is bounded in terms of the derivatives of $E^\star$ of order at most m. We proceed by induction and apply the interior Schauder estimate to obtain bounds for the derivatives of order at most $m+1$ in the cusp. $\square$

We are now ready to introduce a special truncation of the Eisenstein series given a specification of η and the parameters c_* and t as above.

Definition 2. *The special truncation $E^\#$ of the Eisenstein series is given by modification in the cusp regions. For the cusp defining the series where*

$$E(\zeta;2) = (\Im\zeta)^2 + \hat{e}(\zeta),$$

define for $\Im\zeta > 1$ and $\chi = 1 - \eta$,

$$E^\#(\zeta;2) = \chi(-2\pi\Im\zeta - \log c_*)(\Im\zeta)^2 + \chi(-2\pi\Im\zeta + (\log c_*/|t|) + a_0)\hat{e}(\zeta),$$

and for a remaining cusp represented at infinity define for $\Im\zeta > 1$,

$$E^\#(\zeta;2) = \chi(-2\pi\Im\zeta + \log(c_*/|t|) + a_0)E(\zeta;2).$$

It is important that for the change of variables $-2\pi\Im\zeta = \log|z|$, the function $\frac{\partial}{\partial a}\chi(-2\pi\Im\zeta - \log c_*)$ has support in the band $\{e^{a_0}c_* < |z| < c_*\}$ and the function $\frac{\partial}{\partial a}\chi(-2\pi\Im\zeta + \log(c_*/|t|) + a_0)$ has support in the band $\{|t|/c_* < |z| < e^{-a_0}|t|/c_*\}$.

We return to the plumbing construction. Consider R a finite union of Riemann surfaces with punctures and E the Eisenstein series for a cusp p. Consider the family $\{R_t\}$ given by plumbing n pairs of cusps. Each surface R_t, $t = (t_1, \ldots, t_n)$, is obtained by identifying n pairs of subannuli for cusp regions. We first extend the special truncation $E^\#$ by zero on the components of R not containing p. We are ready to define the *melding $E^\dagger$* of Eisenstein series and analyze $(D_{graft} - 2)E^\dagger$. For cusp coordinates z, w of R and constant $c_* < 1$ the punctured discs $\{0 < |z| \le |t|/c_*\}$, $\{0 < |w| \le |t|/c_*\}$ are removed and the annuli $\{|t|/c_* < |z| < c_*\}$, $\{|t|/c_* < |w| < c_*\}$ are identified by the rule $zw = t$ to form a collar. For $z = e^{2\pi i\zeta}$, $\zeta \in \mathbb{H}$, the identified annulus is covered by $\{\log(|t|/c_*) < -2\pi\Im\zeta < \log c_*\}$; the *primary collar band* $\{c_* e^{a_0} \le |z| \le c_*\}$ is covered by the strip $\{\log c_* + a_0 \le -2\pi\Im\zeta \le \log c_*\}$; the *secondary collar band* $\{|t|/c_* \le |z| \le |t|e^{-a_0}/c_*\}$ is covered by the strip $\{\log(|t|/c_*) \le -2\pi\Im\zeta \le \log(|t|/c_*) - a_0\}$. The extended $E^\#$ has support in the z, w cusp regions contained in $\{|z| \ge |t|/c_*\} \cup \{|w| \ge |t|/c_*\}$. We define the *melding $E^\dagger$* on R_t of the Eisenstein series associated to p to be: given on each pair of identified annuli by the sum of values of $E^\#$ at z and at $w = z/t$ on the overlap of $\{|t|/c_* < |z| < c_*\}$ and $\{|t|/c_* < |w| < c_*\}$, and equal to $E^\#$ on the complement of the identified annuli. The melding is a smooth function.

Lemma 3. *Relative to the grafted metric and the prescribed coordinates the melding $E^\dagger$ satisfies*

$$(D_{graft} - 2)E^\dagger = \frac{-1}{4\pi^2}\Lambda + O((-\log|t|)^{-1})$$

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with the remainder term referring to the C^0-norm of a function supported in the collar.

Proof. The grafted metric is given as $dg_t^2 = (ds^2)^{1-\eta}(ds_t^2)^\eta$ for ds^2 the hyperbolic metric of R and ds_t^2 the fiber hyperbolic metric of $\mathcal{V}$. We analyze the contributions of the metrics dg_t^2, ds_t^2 and ds^2 and the contributions of the truncation of the Eisenstein series. On the collar complement the grafted metric coincides with ds^2 and $E^\dagger = E$ is annihilated by the operator. The quantity $(D_{graft} - 2)E^\dagger$ is supported in the collar.

We consider separately the contribution of $E^\#$ on the z and w annuli. We first evaluate $(D_{graft} - 2)E^\#$ on the cusp region defining the series. On the primary and secondary collar bands we have uniformly that $dg_t^2 = ds_t^2(1 + O((\log|t|)^{-2}))$; $D_{graft}E^\#$ can be replaced on the collar bands with $D_t E^\#$ and a remainder. The remainder for the primary band is suitably bounded, and the expression for the secondary band will be considered below. We find with the relations $ds_t^2 = ds_0^2(\Theta \csc \Theta)^2$ for $\Theta = \frac{\pi \log|z|}{\log|t|}$, $z = e^{2\pi i\zeta}$, $\zeta = x + iy$, the hyperbolic Laplacian $D = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, and the expansion from Definition 2 for the truncated Eisenstein series for $\Im\zeta > 1$

$$E^\# = (1 - \eta_1)y^2 + (1 - \eta_2)\hat{e}(\zeta)$$

that

$$(D_t - 2)E^\# = (\Theta^{-1}\sin\Theta)^2 DE^\# - 2E^\# =$$

$$(\Theta^{-1}\sin\Theta)^2\left(2E^\# - 4y^3\frac{\partial\eta_1}{\partial y} - y^4\frac{\partial^2\eta_1}{\partial y^2} - 2y^2\frac{\partial\hat{e}(\zeta)}{\partial y}\frac{\partial\eta_2}{\partial y} - y^2\hat{e}(\zeta)\frac{\partial^2\eta_2}{\partial y^2}\right) - 2E^\#$$

$$= A + B + C$$

for

$$A = ((\Theta^{-1}\sin\Theta)^2 - 1)2E^\#, \quad B = -(\Theta^{-1}\sin\Theta)^2\left(4y^3\frac{\partial\eta_1}{\partial y} + y^4\frac{\partial^2\eta_1}{\partial y^2}\right)$$

and

$$C = -(\Theta^{-1}\sin\Theta)^2\left(2y^2\frac{\partial\hat{e}(\zeta)}{\partial y}\frac{\partial\eta_2}{\partial y} + y^2\hat{e}(\zeta)\frac{\partial^2\eta_2}{\partial y^2}\right).$$

We consider the quantities A, B and C in order. From Lemma 1 $E^\#$ has magnitude $O(y^{-1})$ for y large, and since $-2\pi y = \log|z|$, $\Theta = \frac{\pi \log|z|}{\log|t|}$ then $E^\#$ is bounded as $O((\log|t|)^{-1}\Theta^{-1})$. Since $\Theta^{-1}((\Theta^{-1}\sin\Theta)^2 - 1)$ is bounded for $0 < \Theta < \pi$, it follows that A has magnitude $O((-\log|t|)^{-1})$, as desired. The

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support of B is contained in the primary collar band where $(\Theta^{-1}\sin\Theta)^2 = 1 + O((\log|t|)^{-2})$. Thus the quantity B is given as $-\frac{\partial}{\partial y}(y^4\frac{\partial\eta_1}{\partial y})$ and a suitable remainder. Since $-2\pi y = \log|z|$ it further follows from (4) that B is given as $\frac{-1}{4\pi^2}\Lambda$ and a suitable remainder, as desired. The support of the third term C is contained in the secondary collar band since the derivatives of η_2 are factors. On the support, the difference between $-2\pi y = \log|z|$ and $\log|t|$ is bounded; on the support $\Theta = \pi + O((-\log|t|)^{-1})$, and $(\Theta^{-1}\sin\Theta)^2$ is bounded as $O((\log|t|)^{-2})$. By Lemma 1 on the support of a derivative of η_2 the quantity $(2y^2\frac{\partial\hat{e}}{\partial y} + y^2\hat{e})$ is bounded as $O(-\log|t|)$. The quantity C is bounded as $O((-\log|t|)^{-1})$, as desired.

On a remaining cusp region for identification $E^\# = \hat{e}(\zeta)$ with $\hat{e}(\zeta)$ bounded as $O(y^{-1})$ and in the above notation $(D_t - 2)E^\# = A + C$. The above considerations apply to provide that the evaluation is bounded as $O((-\log|t|)^{-1})$. The melding $E^\dagger$ is given as the sum of values of $E^\#$ for pairs of cusp regions and thus has the desired expansion. $\square$

We continue to consider R a finite union of Riemann surfaces with punctures and $\{R_t\}$ the family given by plumbing n pairs of cusps. For $t = (t_1, \ldots, t_n)$, $|t_k| < c_*^4$, the Riemann surface R_t is independent of the parameter c_*. We will use the intrinsic C^1-norm: for f a differentiable function set $\|f\|_{C^1} = \sup_R|f| + \sup_{\mathbf{T}R}|(ds^2)^{-1/2}df|$. We present the degeneration expansion for the hyperbolic metric.

Theorem 4. *Given a choice of $c_* < 1$ and a cutoff function η, then for all t small the hyperbolic metric ds_{hyp}^2 of R_t has the expansion*

$$ds_{hyp}^2 = dg_t^2\big(1 + \frac{4\pi^4}{3}\sum_{k=1}^n (\log|t_k|)^{-2}(E_{k,1}^\dagger + E_{k,2}^\dagger) + O(\sum_{k=1}^n(\log|t_k|)^{-3})\big).$$

The functions $E_{k,1}^\dagger$ and $E_{k,2}^\dagger$ are the meldings of the Eisenstein series $E(\cdot; 2)$ associated to the pair of cusps plumbed to form the kth collar. The O-term refers to the intrinsic C^1-norm of a function on R_t. The bound depends on the choice of c_*, η and a lower bound for the injectivity radius for the complement of the cusp regions in R.

Proof. The expansion is the combination of Expansion 4.2 [Wol90, pg. 445] with expansion (4) and Lemma 3 above. We follow the approach of the *curvature correction equation* [Wol90, Sec. 4.2]. The approach remains valid for surfaces with punctures and compactly supported curvature perturbations. From equation (4.3) and Lemma 4.1 of [Wol90] $ds_{hyp}^2 = dg_t^2(1 + 2(D_{graft} - $

$2)^{-1}(1 + K_{graft}) + O_k(\|1 + K_{graft}\|^2))$. The remainder terms for (4) and Lemma 3 above are supported in the plumbing collar bands; for a collection of plumbings the combined remainder terms have compact support away from the collars in the plumbed surface. From Estimate A.2 and A.3 of [Wol90, pg.467] the C^1-intrinsic norm of the operator $(D_{graft} - 2)^{-1}$ applied to the remainder terms has norm bounded by the C^0-norm of the remainder terms and vanishes at any additional punctures (the bound does not depend on the injectivity radius of the support). Now combining expansion (4) and Lemma 3 with the uniqueness of solutions of $(D_{graft} - 2)u = v$ with u vanishing at punctures, we find $2(D_{graft} - 2)^{-1}(1 + K_{graft}) = \sum_k \frac{4\pi^4}{3}(\log |t_k|)^{-2}(E_{k,1}^\dagger + E_{k,2}^\dagger + O((-\log |t_k|)^{-1}))$. $\qquad\Box$

Comparison to the infinite energy harmonic map expansion.
For sake of comparison it is instructive to write the metric expansion in terms of the length $\ell = 2\pi^2(-\log |t|)^{-1} + O((\log |t|)^{-2})$, [Wol90, Ex. 4.3], of the geodesic of the ds_t^2 fiber metric of $\mathcal{V}$

$$ds_{hyp}^2 = dg_t^2\big(1 + \sum_k \frac{\ell_k^2}{3}(E_{k,1}^\dagger + E_{k,2}^\dagger) + O(\sum_k \ell_k^3)\big).$$

M. Wolf's infinite energy harmonic map parameterization of the plumbing family provides that

$$ds_{hyp}^2 = ds^2 + \sum_k \ell_k^2\big(\frac{1}{2}\Re(\Phi_{k,1} + \Phi_{k,2}) + \frac{1}{6}(E_{k,1} + E_{k,2})ds^2\big) + O(\sum_k \ell_k^3),$$

where $\Phi_{k,1}$ and $\Phi_{k,2}$ are holomorphic quadratic differentials with unit *residue* for the cusps paired for the kth-node [Wol91, esp. Cor. 5.4]. The two expansions are similar in spite of the different contexts. The grafting expansion is given in terms of a metric on each surface R_t and the comparison between metrics is by conformal inclusion maps. Wolf's expansion is in terms of a family of metrics on R and the comparison between fibers is by infinite energy (non conformal) harmonic maps. The appearance of holomorphic quadratic differentials is attributed to a difference of gauge. We found in [Wol94] for a closed geodesic α of R and Φ the holomorphic wt. 4 Eisenstein series (the $\Gamma_\mathbb{Z}$-coset sum of $(d\zeta)^2$ for the cusp represented at infinity) that

$$\int_\alpha E(\zeta; 2)ds = 3\int_\alpha \Phi(ds)^{-1}.$$

Each approach for the expansion gives for the length of a closed geodesic

disjoint from plumbing collars

$$\ell_\alpha(\ell) = \ell_\alpha(0) + \sum_k \frac{\ell_k^2}{6} \int_\alpha (E_{k,1} + E_{k,2})ds + O(\sum_k \ell_k^3),$$

in agreement with [Wol94, formula (1.9)]. The positivity of the variation of length is a basic property of degeneration; see also expansion (3). The positive order 2 variation in ℓ corresponds to the positive remainder term for the pairings of WP gradients of length functions, [Wolom].

4   The WP degeneration expansion

We apply the expansion for the hyperbolic metric to develop a degeneration expansion for the WP metric. We identify the leading term, a general remainder term third-order in distance, and for directions parallel to the divisor of nodal curves we identify the TZ metric as the fourth-order term in distance. We first describe *local coordinates* for the deformation space of a Riemann surface with nodes and punctures. We employ a modification of the standard coordinates [Ber74, Mas76] and combine the formulation and the expansion for the hyperbolic metric to present a refined form of Masur's expansion of the WP metric.

   We continue with discussion of the plumbing variety $\mathcal{V}$ and consider infinitesimal variations of the fibers. To illustrate a general point we first take $c_* = 1$ and consider a fiber $\{|t| \leq |z| \leq 1\}$. The parameter t is a boundary point of the annulus. The boundary points t, 1 will be included in the data for gluings. To describe the variation of annuli with boundary points, we now specify a quasiconformal map ζ from the pointed t-annulus to the pointed t'-annulus $\zeta(z) = zr^{\beta(r,t')}$, $z = re^{i\theta}$, with $\frac{\partial\beta}{\partial r}$ compactly supported in the annulus. The boundary conditions are $\zeta(1) = 1$, and by specification $t|t|^{\beta(|t|,t')} = t'$. On differentiating in t' and evaluating at $(|t|,t)$ we find the boundary condition $t \log|t| \, \dot\beta(|t|,t) = 1$. The infinitesimal variation of the map is the vector field $\dot\zeta(z) = z \log r \, \dot\beta(r,t)$ for $\dot\zeta, \dot\beta$ the first t-derivatives. The map ζ varies from the identity and has Beltrami differential

$$\bar\partial\dot\zeta = \frac{z}{2\bar z} \frac{\partial}{\partial \log r}(\dot\beta(r,t) \log r) \frac{\overline{dz}}{dz}. \tag{5}$$

For sake of later application we evaluate the pairing with a quadratic differ-

ential $z^\alpha \left(\frac{dz}{z}\right)^2$,

$$\int_{\{|t|<|z|<1\}} \bar{\partial} \dot\zeta \, z^\alpha \left(\frac{1}{z}\right)^2 d\mathcal{E} = \int_{\{|t|<|z|<1\}} \frac{z^\alpha}{2z\bar{z}} \frac{\partial}{\partial \log r} (\dot\beta \log r) \, d\mathcal{E}$$

where for $\alpha = 0$, then

$$= \pi\dot\beta \log r \Big|_{|t|}^{1} = \frac{-\pi}{t}, \tag{6}$$

and otherwise, then

$$= 0,$$

for $d\mathcal{E}$ the Euclidean area element and where we have applied the boundary condition for $\dot\beta$; the evaluation involves fixing a normalization for the Serre duality pairing and agrees with [Mas76, Prop. 7.1].

We recall the description of families of *Riemann surfaces with nodes* [Ber74, Mas76, Wol90]. A Riemann surface R with nodes and punctures is a connected complex space, such that every point has a neighborhood isomorphic to either the unit disc in $\mathbb{C}$, the punctured unit disc in $\mathbb{C}$, or the germ at the origin in $\mathbb{C}^2$ of the union of the coordinate axes. The points with punctured disc neighborhoods are *primary punctures.* R is *stable* provided each component of $R - \{nodes\}$ has negative Euler characteristic, i.e. has a hyperbolic metric. A regular k-differential on R is the assignment of a meromorphic k-differential Θ_j for each component R_j of $R - \{nodes\}$ such that: i) each Θ_* has poles only at the punctures of R_* with orders at most k, ii) poles at primary punctures have order at most $k-1$, and iii) if punctures p, q are paired to form a node then $Res_p\Theta_* = (-1)^k Res_q\Theta_*$ [Ber74].

We recall the deformation theory of Riemann surfaces with punctures and then with nodes. A deformation neighborhood of the marked surface R is given by specifying smooth Beltrami differentials $\nu_1, \ldots, \nu_m$ spanning the Dolbeault group $H^{0,1}_{\bar\partial}(\overline{R}, \mathcal{E}((\kappa p_1 \cdots p_d)^{-1}))$ for κ the canonical bundle of $\overline{R}$ (the closure of R) and $p_1, \ldots, p_d$ the point line bundles for the punctures [Kod86]. For $s \in \mathbb{C}^m$ set $\nu(s) = \sum_k s_k\nu_k$; for s small there is a Riemann surface $R^{\nu(s)}$ and a diffeomorphism $\zeta : R \to R^{\nu(s)}$ satisfying $\bar\partial\zeta = \nu(s)\partial\zeta$. The family of surfaces $\{R^{\nu(s)}\}$ represents a neighborhood of the marked Riemann surface in its Teichmüller space. The Beltrami differentials can be modified a small amount so that in terms of each *cusp coordinate* the diffeomorphisms $\zeta^{\hat\nu(s)}$ are simply rotations [Wol92, Lemma 1.1]; $\zeta^{\hat\nu(s)}$ is a hyperbolic isometry in a neighborhood of the cusps; $\zeta^{\hat\nu(s)}$ cannot be complex analytic in s, but is real analytic. We note that for s small the s-derivatives of $\nu(s)$ and $\hat\nu(s)$ are close. We say that $\zeta^{\hat\nu(s)}$ *preserves cusp coordinates*. The

parameterization provides a key ingredient for obtaining simplified estimates of the degeneration of hyperbolic metrics and an improved expansion for the WP metric.

We describe a *local manifold cover* of the compactified moduli space $\overline{\mathcal{M}}$. For R having nodes and punctures, $R_0 = R - \{nodes\}$ is a union of Riemann surfaces with punctures. The quasiconformal deformation space of R_0, $Def(R_0)$, is the product of the Teichmüller spaces of the components of R_0. As above for $m = \dim Def(R_0)$ there is a real analytic family of Beltrami differentials $\hat{\nu}(s)$, s in a neighborhood of the origin in $\mathbb{C}^m$, such that $s \to R_s = R^{\hat{\nu}(s)}$ is a coordinate parameterization of a neighborhood of R_0 in $Def(R)$ and the prescribed mappings $\zeta^{\hat{\nu}(s)} : R_0 \to R^{\hat{\nu}(s)}$ preserve the cusp coordinates at each puncture. For R with n nodes we prescribe the plumbing data $(U_k, V_k, z_k, w_k, t_k)$, $k = 1, \ldots, n$, for $R^{\hat{\nu}(s)}$, where z_k on a neighborhood U_k and w_k on a neighborhood V_k are cusp coordinates relative to the $R^{\hat{\nu}(s)}$-hyperbolic metric (the plumbing data varies with s). The parameter t_k parameterizes opening the kth node. For all t_k suitably small, perform the n prescribed plumbings to obtain the family $R_{s,t} = R^{\hat{\nu}(s)}_{t_1,\ldots,t_n}$. The tuple $(s,t) = (s_1, \ldots, s_m, t_1, \ldots, t_n)$ provides real analytic local coordinates, the *hyperbolic metric plumbing coordinates*, for the local manifold cover of $\overline{\mathcal{M}}$ at R, [Mas76, Wol88] and [Wol90, Secs. 2.3,2.4]. The coordinates have a special property: for s fixed the parameterization is holomorphic in t. The property is a basic feature of the plumbing construction. The family $R_{s,t}$ parameterizes the small deformations of the marked noded surface with punctures R.

We review the geometry of the local manifold covers. For a complex manifold M the complexification $\boldsymbol{T}^{\mathbb{C}} M$ of the $\mathbb{R}$-tangent bundle is decomposed into the subspaces of *holomorphic* and *antiholomorphic* tangent vectors. A Hermitian metric g is prescribed on the holomorphic subspace. For a general complex parameterization $s = u + iv$ the coordinate $\mathbb{R}$-tangents are expressed as $\frac{\partial}{\partial u} = \frac{\partial}{\partial s} + \frac{\partial}{\partial \bar{s}}$ and $\frac{\partial}{\partial v} = i\frac{\partial}{\partial s} - i\frac{\partial}{\partial \bar{s}}$. For the $R_{s,t}$ parameterization the s-parameters are not holomorphic while for s-parameters fixed the t-parameters are holomorphic; $\{\frac{\partial}{\partial s_j} + \frac{\partial}{\partial \bar{s}_j}, i\frac{\partial}{\partial s_j} - i\frac{\partial}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}, i\frac{\partial}{\partial t_k}\}$ is a basis over $\mathbb{R}$ for the tangent space of the local manifold cover. For a smooth Riemann surface the dual of the space of holomorphic tangents is the space of quadratic differentials with at most simple poles at punctures. The following is a modification of Masur's result [Mas76, Prop. 7.1].

Lemma 5. *The hyperbolic metric plumbing coordinates (s,t) are real analytic and for s fixed the parameterization is holomorphic in t. Provided the modification $\hat{\nu}$ is small, for a neighborhood of the origin there are families*

in (s,t) of regular 2-differentials φ_j, ψ_j, j=1,...,m and η_k, k=1,...,n such that:

1. *Each regular 2-differential has an expansion of the form $\varphi(s,t) = \varphi(s,0) + O(t)$ locally away from the nodes of R.*

2. *For $R_{s,t}$ with $t_k \neq 0$, all k, $\{\varphi_j, \psi_j, \eta_k, i\eta_k\}$ forms the dual basis to $\{\frac{\partial\hat{\nu}(s)}{\partial s_j} + \frac{\partial\hat{\nu}(s)}{\partial \bar{s}_j}, i\frac{\partial\hat{\nu}(s)}{\partial s_j} - i\frac{\partial\hat{\nu}(s)}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}, i\frac{\partial}{\partial t_k}\}$ over $\mathbb{R}$.*

3. *For $R_{s,t}$ with $t_k = 0$, all k, the η_k, k=1,...,n, are trivial and the $\{\varphi_j, \psi_j\}$ span the dual of the holomorphic subspace $\boldsymbol{T}Def(R_0)$.*

Proof. The setting compares to that considered by Masur. The new element: the variation of the plumbing data with s is prescribed by a Schiffer variation for a gluing-function real analytically depending on the parameter s [Wol88, pg.410]. As noted for s fixed, plumbing produces a holomorphic family. Following Masur the families of regular 2-differentials $\{\varphi_j, \psi_j, \eta_k\}$ are obtained by starting with a local frame $\mathcal{F}$ of regular 2-differentials and prescribing the pairings with $\{\frac{\partial\hat{\nu}}{\partial s_j}, \frac{\partial\hat{\nu}}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}\}$ [Mas76, Sec. 5 and Prop. 7.1]. At an initial point the basis is simply given by a linear transformation of the frame $\mathcal{F}$. Also the elements of $\mathcal{F}$ each have an expansion of the desired form in t. The prescribed basis will exist in a neighborhood provided the pairings are continuous with the desired expansion in t. We first consider the pairings with $\frac{\partial}{\partial t_k}$. From (5) we have the Beltrami differential for the pairing with $\frac{\partial}{\partial t_k}$, $k = 1, \ldots, n$. In particular for a plumbing collar of $R_{s,t}$ let z (or w) be the coordinate of the plumbing. A quadratic differential φ on $R_{s,t}$ can be factored on the collar into a product of $\left(\frac{dz}{z}\right)^2$ and a function holomorphic in z. We write $\mathcal{C}_k(\varphi)$ for the constant coefficient of the Laurent expansion of the function factor. From (6) the pairing with $\frac{\partial}{\partial t_k}$ is the linear functional $-\frac{\pi}{t_k}\mathcal{C}_k$. From Masur's considerations [Mas76, Sec. 5, esp. 5.4, 5.5] the pairing of $\frac{\partial}{\partial t_k}$ with the local frame $\mathcal{F}$ is continuous with the desired expansion, and there are regular 2-differentials $\{\varphi_j, \psi_j, \eta_k^*\}$ with: $\mathcal{C}_\ell(\varphi_j) = \mathcal{C}_\ell(\psi_j) = 0$, $j = 1, \ldots, m$; $\mathcal{C}_\ell(\eta_k^*) = \delta_{k\ell}$, $k, \ell = 1, \ldots, n$. The 2-differentials $\eta_k = \frac{-t_k}{\pi}\eta_k^*$, $k = 1, \ldots, n$ have the desired pairings with $\frac{\partial}{\partial t_k}$ and the desired expansion. The final matter is to note that the pairings of $\{\varphi_j, \psi_j, \eta_k^*\}$ with $\{\frac{\partial\hat{\nu}}{\partial s_\ell}, \frac{\partial\hat{\nu}}{\partial \bar{s}_\ell}\}$ are continuous in (s,t). By construction the differential $\hat{\nu}(s)$ is supported in the complement of the plumbing collars [Wol92, Lemma 1.1]. On the support of $\hat{\nu}(s)$ the 2-differentials are real analytic in (s,t). The pairings are continuous and even real analytic. $\square$

We note general matters for consideration below. As above, for z a plumbing collar coordinate for $R_{s,t}$, a quadratic differential ψ can be factored

on the collar as the product of $\left(\frac{dz}{z}\right)^2$ and a holomorphic function. The value $\mathcal{C}(\psi)$ is the constant coefficient of the Laurent expansion of the function. The surface $R_{s,t}$ is constructed by plumbing $\left(R_s\right)^*_{c_*^2}$ with the R_s-hyperbolic cusp coordinates. The surface $R_{s,t}$ is the *disjoint union* of $\left(R_s\right)^*_{c_*}$, R_s with the cuspidal discs $|z_*|, |w_*| < c_*$ removed, and annuli $\{|t|/c_* < |z| < c_*\}$.

We further note a general matter, the metric-dual map for a Hermitian product space. For a vector space V with Hermitian product $\langle\ ,\ \rangle$, a complex conjugate-linear isomorphism to the dual space V^* is defined by the rule $v \in V$ is mapped to the linear functional $\delta(v) = \langle\ , v \rangle \in V^*$. The dual Hermitian product $\langle\langle\ ,\ \rangle\rangle$ for V^* is determined for $v, w \in V$ by $\langle\langle \delta(v), \delta(w) \rangle\rangle = \langle w, v \rangle$. For $\{v_j\}$ a basis (over $\mathbb{R}$ or $\mathbb{C}$) for V the dual basis for V^* can be expressed in terms of δ. For the Hermitian pairing matrix $(g_{j\ell}) = (\langle v_j, v_\ell \rangle)$ and the inverse matrix $(g^{j\ell})$, then $\{\delta(\Sigma_\ell g^{j\ell} v_\ell)\}$ is the corresponding dual basis for V^*. In particular the evaluation of v_k with $\delta(\Sigma_\ell g^{j\ell} v_\ell)$ is $\Sigma_\ell \overline{g^{j\ell}} g_{k\ell} = \delta_{kj}$. For the dual basis $\{v_j^*\}$ of V^* the pairing matrix has entries $\langle\langle v_j^*, v_k^* \rangle\rangle = \langle\langle \delta(\Sigma_\ell g^{j\ell} v_\ell), \delta(\Sigma_m g^{km} v_m) \rangle\rangle = \Sigma_{\ell,m} g^{km} g_{m\ell} \overline{g^{j\ell}} = g^{kj}$. The pairing matrix for the dual metric is the conjugate inverse of the original pairing matrix.

We note the particulars for Teichmüller space and the WP metric. At a marked Riemann surface R the tangent space of Teichmüller space is naturally identified with $\mathcal{H}(R)$, the space of Beltrami differentials harmonic relative to the R hyperbolic metric ds^2 [Ahl61]. An element μ of $\mathcal{H}(R)$ is given as $\mu = \overline{\varphi}(ds^2)^{-1}$ for $\varphi = \overline{\mu}(ds^2) \in Q(R)$, an integrable holomorphic quadratic differential. The pairing of $\mathcal{H}(R)$ and $Q(R)$ is given by integration. For $\varphi, \psi \in Q(R)$ the integral $\int_R \psi \overline{\varphi}(ds^2)^{-1}$ is both the cotangent-tangent pairing of ψ and $\overline{\varphi}(ds^2)^{-1}$ and the Hermitian pairing of ψ and φ. The mapping $\varphi \rightarrow \overline{\varphi}(ds^2)^{-1}$ is the conjugate-linear metric-dual mapping from $Q(R)$ to $\mathcal{H}(R)$. For $\{\mu_j\}$ a basis for $\mathcal{H}(R)$, $\{\kappa_j\}$ the dual basis for $Q(R)$, and the pairing matrix $(g_{j\ell}) = (\langle \kappa_j, \kappa_\ell \rangle_{WP})$, the bases are related by $\overline{\Sigma_\ell g^{j\ell} \kappa_\ell} = \mu_j(ds^2)$.

We now apply the two-term expansion of Theorem 4 for the hyperbolic metric to obtain a two-term expansion for the WP metric. The metric expansion will be substituted for the original construction of [Mas76, Sec. 6] to obtain a refinement of the original expansion. In the following the Eisenstein series $E_{k,1}$ and $E_{k,2}$ are for the pair of cusps representing the kth node.

Theorem 6. *For a noded Riemann surface R with punctures the hyperbolic metric plumbing coordinates for $R_{s,t}$ provide real analytic coordinates for a local manifold cover neighborhood for $\overline{\mathcal{M}}$. The parameterization is holomorphic in t for s fixed. On the local manifold cover the WP metric is formally*

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Hermitian satisfying:

1. *For $t_k = 0$, $k = 1, \ldots, n$, the restriction of the metric is a smooth Kähler metric, isometric to the WP product metric for a product of Teichmüller spaces.*

2. *For the tangents $\{\frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}, \frac{\partial}{\partial t_k}\}$ and the quantity $\sigma = \sum_{k=1}^{n} (\log |t_k|)^{-2}$*

   then: $g_{WP}(\frac{\partial}{\partial t_k}, \frac{\partial}{\partial t_k})(s, t) = \dfrac{\pi^3}{|t_k|^2(-\log^3 |t_k|)}(1 + O(\sigma))$;

   $g_{WP}(\frac{\partial}{\partial t_k}, \frac{\partial}{\partial t_\ell})$ is $O((|t_k t_\ell| \log^3 |t_k| \log^3 |t_\ell|)^{-1})$ for $k \neq \ell$;

   and for the tangents $\mathfrak{u} = \frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}$:

   $g_{WP}(\frac{\partial}{\partial t_k}, \mathfrak{u})$ is $O((|t_k|(-\log^3 |t_k|))^{-1})$.

3. *For $\mathfrak{u} = \frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}$, represented at $R_{s,0}$ by μ_j and $\mathfrak{v} = \frac{\partial}{\partial s_\ell}, \frac{\partial}{\partial \bar{s}_\ell}$ represented at $R_{s,0}$ by μ_ℓ then:*

   $g_{WP}(\mathfrak{u}, \mathfrak{v})(s, t) =$

   $g_{WP}(\mathfrak{u}, \mathfrak{v})(s, 0) + \frac{4\pi^4}{3} \sum_{k=1}^{n} (\log |t_k|)^{-2} \langle \mu_j, \mu_\ell(E_{k,1} + E_{k,2}) \rangle_{WP}(s, 0)$

   $+ O(\sum_{k=1}^{n} (-\log |t_k|)^{-3})$.

Proof. We develop the expansion of the dual metric for the basis provided in Lemma 5. In order to consider integrals we decompose $R_{s,t}$ into $(R_s)^*_{c_*}$ and the collars $\{|t|/c_* \leq |z| \leq c_*\}$. We compute principal terms of expansions and gather remainder terms. From Lemma 5 the variation of the 2-differentials φ_j, ψ_j, η_k is a remainder term. From Theorem 4 for the principal term for the hyperbolic metric we have $dg_t^2(1 + \frac{4\pi^4}{3} \sum(\log |t_k|)^{-2}(E_{k,1} + E_{k,2}))$ on $(R_s)^*_{c_*}$ and ds_t^2 the plumbing variety metric on the collars. On a collar $\{|t|/c_* \leq |z| \leq c_*\} = \{|t|/c_* \leq |w| \leq c_*\}$ a quadratic differential is considered as the product of $\left(\frac{dz}{z}\right)^2 = \left(\frac{dw}{w}\right)^2$ and a function factor.

   The first term. We begin with a model calculation for the hyperbolic metric on the plumbing variety

$$\int_{\{|t|/c_* < |z| < c_*\}} |z^\alpha (\frac{dz}{z})^2|^2 \, (ds_t^2)^{-1} =$$

$$\frac{2}{\pi} \int_{|t|/c_*}^{c_*} \left(\log |t| \sin \frac{\pi \log r}{\log |t|} \right)^2 r^{2\alpha} \, d \log r$$

where for $\alpha = 0$, $\mu = \dfrac{\log r}{\log |t|}$ and $\epsilon = \dfrac{\log c_*}{\log |t|}$, then

$$= \frac{2}{\pi}(-\log^3 |t|) \int_\epsilon^{1-\epsilon} \sin^2 \pi\mu \, d\mu = \frac{1}{\pi}(-\log^3 |t|) + O(1),$$

and for $\alpha = 1$, since $|\sin \mu| \leq |\mu|$, then
$$= O(1).$$

We first consider the contribution to the pairing from the collars. Consider the contribution for the ℓth collar. By construction η_ℓ is the unique quadratic differential from the dual basis with a nonzero $\mathcal{C}_\ell$ evaluation; as above $\eta_\ell = -\frac{t_\ell}{\pi}\eta_\ell^*$. In particular $\mathcal{C}_\ell(\eta_\ell^*) = 1$ and as above the contribution to the self pairing for η_ℓ^* for the collar is $\frac{1}{\pi}(-\log^3 |t_\ell|) + O(1)$. In general a quadratic differential on a collar is factored as $\left(\frac{dz}{z}\right)^2 (f_z + \boldsymbol{c} + f_w)$ for f_z holomorphic in $|z| < c$, $f_z(0) = 0$; $\boldsymbol{c}$ the $\mathcal{C}$-evaluation value and f_w holomorphic in $|w| < c$, $f_w(0) = 0$. Furthermore f_z, resp. f_w, is given as the Cauchy integral of f over $|z| = c_*$, resp. $|w| = c_*$. From the Schwarz Lemma $|f_z| \leq c'|z| \max_{|z|=c_*} |f|$ with a corresponding bound for $|f_w|$. The 2-differentials φ_j, ψ_j, η_k^* on $|z| = c_* + \epsilon_0$ and $|w| = c_* + \epsilon_0$ depending continuously on (s, t) and so admit decompositions with f_z and f_w uniformly bounded. The uniform bounds are combined with the majorant bound $|\sin \mu| \leq |\mu|$ to bound integrands for the collars.

We combine the above considerations and substitute the grafted metric dg_t^2 for the metric principal term to obtain the expansions

$$\langle \eta_k^*, \eta_k^* \rangle_{WP} = \frac{1}{\pi}(-\log^3 |t_k|)\,(1 + O(\sum_{\ell=1}^{n} (\log |t_\ell|)^{-2})),$$

$$\langle \eta_k^*, \eta_\ell^* \rangle_{WP} = O(1) \text{ for } k \neq \ell,$$

and for $\mathfrak{a} = \varphi_j, \psi_j$; $\mathfrak{b} = \varphi_\ell, \psi_\ell$:

$$\langle \mathfrak{a}, \eta_k^* \rangle_{WP} = O(1).$$

The pairing $\langle \mathfrak{a}, \mathfrak{b} \rangle_{WP}$ will be considered in the following paragraphs.

The second term. We are ready to determine the second-term in the expansion for $\langle \mathfrak{a}, \mathfrak{b}\rangle_{WP}(s,t)$ with $\mathfrak{a} = \varphi_j, \psi_j$; $\mathfrak{b} = \varphi_\ell, \psi_\ell$ and the contribution of the Eisenstein series. From Lemma 5 and Theorem 4 a main step is to give an expansion in t for the integrals of $\mathfrak{a}\overline{\mathfrak{b}}(dg_t^2)^{-1}$ and $\mathfrak{a}\overline{\mathfrak{b}}(E_{k,1}^\dagger + E_{k,2}^\dagger)(dg_t^2)^{-1}$ over a collar $\{|t|/c_* \leq |z|, |w| \leq c_*\}$. The goal is expansions with remainders at least $O((-\log|t|)^{-3})$ and principal terms with remainders $O(c_*)$ for c_* tending to zero.

The expansion for $\mathfrak{a}\overline{\mathfrak{b}}$ is the first consideration. The holomorphic quadratic differentials have vanishing $\mathcal{C}$-evaluations and thus on the collars have factorizations $\mathfrak{a} = (\frac{dz}{z})^2(\mathfrak{a}_z + \mathfrak{a}_w)$ and $\mathfrak{b} = (\frac{dz}{z})^2(\mathfrak{b}_z + \mathfrak{b}_w)$. As above, the summands are bounded as $|\mathfrak{a}_z|, |\mathfrak{b}_z| \leq |z|c'$ and $|\mathfrak{a}_w|, |\mathfrak{b}_w| \leq |w|c'$ for a suitable constant depending on $\mathfrak{a}$ and $\mathfrak{b}$. Now from expansion (2), the bound $|\sin\mu| \leq 1$ and the relation $zw = t$ we have that $(|\mathfrak{a}_z\overline{\mathfrak{b}_w}| + |\mathfrak{a}_w\overline{\mathfrak{b}_z}|)|\frac{dz}{z}|^4(dg_t^2)^{-1} \leq 2c'^2|t|(\log|t|)^2|\frac{dz}{z}|^2$. Furthermore the meldings $E_{k,*}^\dagger$ are uniformly bounded in t. It follows that the integrals of $\mathfrak{a}_z\overline{\mathfrak{b}_w}$ and $\mathfrak{a}_w\overline{\mathfrak{b}_z}$ over a collar are bounded by $O(|t|(-\log|t|)^3)$ with a similar bound on including the meldings. The next matter is the contribution of $\mathfrak{a}_z\overline{\mathfrak{b}_z}$ and $\mathfrak{a}_w\overline{\mathfrak{b}_w}$; we illustrate by considering the former. The expansion in t of $\mathfrak{a}$ and $\mathfrak{b}$ is the first consideration. The summands $\mathfrak{a}_z$ and $\mathfrak{b}_z$ vanish at the origin and are holomorphic in t. The differences $\mathfrak{a}_z(t) - \mathfrak{a}_z(0)$ and $\mathfrak{b}_z(t) - \mathfrak{b}_z(0)$ are bounded by the Schwarz Lemma by $|zt| \leq c'$, again for a suitable constant depending on $\mathfrak{a}$ and $\mathfrak{b}$. It follows that $|\mathfrak{a}_z(t)\overline{\mathfrak{b}_z(t)}||\frac{dz}{z}|^4(dg_t^2)^{-1} = |\mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}||\frac{dz}{z}|^4(dg_t^2)^{-1} + O(c'^2|t|(\log|t|)^2|dz|^2)$, again with a similar bound on including the meldings. The integral of the remainder term over the collar is $O(|t|(\log|t|)^2)$. Overall the z-term contribution of $\mathfrak{a}\overline{\mathfrak{b}}$ has integrand principal term $\mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}|\frac{dz}{z}|^4(dg_t^2)^{-1}$ and remainder term $O(|t|(-\log|t|)^3)$.

The expansion in t of $(dg_t^2)^{-1}$ is the next consideration. We proceed by separately considering the subdomains $\{|t|^{1/2} \leq |z| \leq c_*\}$ and $\{|t|/c_* \leq |z| \leq |t|^{1/2}\}$. We present an expansion for the first subdomain and an overall bound for the second subdomain. On the interval $[0, \frac{\pi}{2}]$ we have that $0 \leq \mu - \frac{1}{6}\mu^3 \leq \sin\mu \leq \mu$ and thus $0 \leq \mu^2 - \sin^2\mu \leq \frac{1}{3}\mu^4 - \frac{1}{36}\mu^6$. We have in general from (3) that $ds_0^2 \leq dg_t^2 \leq ds_t^2$. It follows for the first subdomain (provided $c_* \leq e^{-1}$) that the quantity $(\log|z|)^2 - |\frac{dz}{z}|^2(dg_t^2)^{-1}$ is bounded by $O((\log|t|)^{-2}(\log|z|)^6)$. On the first domain we now have the expansion $\mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}|\frac{dz}{z}|^4(dg_t^2)^{-1} = \mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}|\frac{dz}{z}|^4(ds_0^2)^{-1} + O(c'^2(\log|t|)^{-2}(\log|z|)^6|dz|^2)$, again with a similar bound on including the meldings. The integral of the remainder term over the first subdomain is $O(c_*(\log|t|)^{-2})$. We are ready to consider the second subdomain $\{|t|/c_* \leq |z| \leq |t|^{1/2}\}$ where $|\frac{dz}{z}|^2(dg_t^2)^{-1}$ is bounded by $(\log|t|)^2$ and the integrand $\mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}|\frac{dz}{z}|^4(dg_t^2)^{-1}$ is bounded

by $c'^2(\log|t|)^2|dz|^2$. The integral over the second subdomain is bounded by $O(|t|(\log|t|)^2)$, again with a similar bound on including the meldings.

We are ready to gather our considerations and present the expansions for integrals over a collar

$$\int_{\{|t|/c_* \le |z| \le c_*\}} \mathfrak{a}\overline{\mathfrak{b}}(dg_t^2)^{-1}$$
$$= \int_{\{|t|^{1/2} \le |z| \le c_*\}} \mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}\big|\frac{dz}{z}\big|^4(ds_0^2)^{-1}$$
$$+ \int_{\{|t|^{1/2} \le |w| \le c_*\}} \mathfrak{a}_w(0)\overline{\mathfrak{b}_w(0)}\big|\frac{dw}{w}\big|^4(ds_0^2)^{-1}$$
$$+ \ O(c_*(\log|t|)^{-2}) \ + \ O(|t|(-\log|t|)^3)$$

and

$$\int_{\{|t|/c_* \le |z| \le c_*\}} \mathfrak{a}\overline{\mathfrak{b}}(E_{k,1}^\dagger + E_{k,2}^\dagger)(dg_t^2)^{-1}$$
$$= \int_{\{|t|^{1/2} \le |z| \le c_*\}} \mathfrak{a}_z(0)\overline{\mathfrak{b}_z(0)}(E_{k,1}^\dagger + E_{k,2}^\dagger)\big|\frac{dz}{z}\big|^4(ds_0^2)^{-1}$$
$$+ \int_{\{|t|^{1/2} \le |w| \le c_*\}} \mathfrak{a}_w(0)\overline{\mathfrak{b}_w(0)}(E_{k,1}^\dagger + E_{k,2}^\dagger)\big|\frac{dw}{w}\big|^4(ds_0^2)^{-1} + \ O((\log|t|)^{-2}).$$

Observations are in order. The difference between a melding $E^\dagger$ and an Eisenstein series E is bounded by $c(\log|z|)^2$. The meldings can now be replaced by Eisenstein series with the additional contributions bounded by $O(c_*)$. The resulting integrals on the right hand side over $\{|t|^{1/2} \le |z|, |w| \le c_*\}$ can now be extended to integrals over the discs $\{|z|, |w| \le c_*\}$ with the additional contributions bounded by $O(|t|(\log|t|)^4)$. The resulting integrals on the right hand side are now for functions on domains in $R_{s,0}$.

The dual metric expansion. We now combine the expansions for the integrals over collars with Theorem 4 to find the expansion for the remaining pairing

$$\langle \mathfrak{a}, \mathfrak{b}\rangle_{WP}(s,t) = \langle \mathfrak{a}, \mathfrak{b}\rangle_{WP}(s,0)$$
$$- \frac{4\pi^4}{3}\sum_{k=1}^n (\log|t_k|)^{-2}\big(\langle \mathfrak{a}, \mathfrak{b}(E_{k,1} + E_{k,2})\rangle_{WP}(s,0) + O(c_*)\big)$$
$$+ O(\sum_{k=1}^n (-\log|t_k|)^{-3}).$$

Observations are again in order. The coefficients of $(\log |t_k|)^{-2}$ are independent of c_* since the Hermitian products $\langle \mathfrak{a}, \mathfrak{b} \rangle_{WP}$ are independent of c_* and the remainder term has strictly smaller order of magnitude in t. It follows that the $O(c_*)$ terms vanish.

The metric expansion. The final matter is to relate the expansion for the dual metric to the expansion for the metric. The first term is an immediate consequence of the following Lemma. We consider the second term and denote by V the span of the cotangent fields $\{ds_j, d\bar{s}_j, ds_\ell, d\bar{s}_\ell\}$. For a vector space V with basis $\{v_j\}$ and a family of Hermitian pairings $\langle \, , \, \rangle_\epsilon$, let $A_\epsilon = (\langle v_j, v_\ell \rangle_\epsilon)$ be the variation of the pairing matrix. For a variation $A_\epsilon = A + \epsilon B$, the inverse matrix satisfies $(A + \epsilon B)^{-1} = A^{-1} - \epsilon A^{-1} B A^{-1} + O(\epsilon^2) = A^{-1} - \epsilon A^{-1} B \overline{(A^t)^{-1}} + O(\epsilon^2)$. As already noted the pairing matrix for the dual metric is given by the conjugate inverse matrix. We apply the considerations for $\{\kappa_\ell\}$ the basis for $Q(R)$ with pairing matrix $g_{j\ell} = (\langle \kappa_j, \kappa_\ell \rangle_{WP})$ to find the pairing for the dual basis $\mu_j(ds_0^2) = \overline{\Sigma_\ell g^{j\ell} \kappa_\ell}$ for $\{\mu_j\}$ the dual basis for $\mathcal{H}(R)$. In conclusion for $\mathfrak{u} = \frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}$ represented at $R_{s,0}$ by μ_j, and $\mathfrak{v} = \frac{\partial}{\partial s_\ell}, \frac{\partial}{\partial \bar{s}_\ell}$ represented at $R_{s,0}$ by μ_ℓ we have the expansion

$$
g_{WP}(\mathfrak{u}, \mathfrak{v})(s, t)
$$

$$
= g_{WP}(\mathfrak{u}, \mathfrak{v})(s, 0) + \frac{4\pi^4}{3} \sum_{k=1}^{n} (\log |t_k|)^{-2} \langle \mu_j, \mu_\ell(E_{k,1} + E_{k,2}) \rangle_{WP}(s, 0)
$$

$$
+ O\left(\sum_{k=1}^{n} (-\log |t_k|)^{-3} \right).
$$

$\square$

For $\mathcal{A}$ a symmetric $m + n \times m + n$ matrix

$$
\begin{pmatrix}
\lambda_1 & \cdots & a_{\ell j} & \cdots \\
\vdots & \lambda_k & \vdots & \vdots \\
a_{j\ell} & \cdots & \ddots & \cdots \\
\cdots & \cdots & \cdots & B
\end{pmatrix}
$$

with λ_k, $1 \leq k \leq n$;
$a_{j\ell}$, $1 \leq j \leq m + n$, $j \neq \ell$, $1 \leq \ell \leq n$ and
$B = (b_{j\ell})$ a symmetric $m \times m$ matrix,

we consider the situation that $\lambda_1, \ldots, \lambda_n$ are large compared to the $a_{j\ell}$ and $b_{j\ell}$. We have the following expansion presented in [Wol03, Prop. 3].

Lemma 7. *For $\det B \neq 0$, and $\rho = \sum_{k=1}^{n} \lambda_k^{-1}$ then:*

$$\det \mathcal{A} = \det B \prod_{k=1}^{n} \lambda_k (1 + O(\rho))$$

and $\mathcal{A}^{-1} = (\alpha_{j\ell})$ where: for $1 \leq k \leq n$, $\alpha_{kk} = \lambda_k^{-1}(1 + O(\rho))$; for $1 \leq j < \ell \leq n$, $\alpha_{j\ell}$ is $O((\lambda_j \lambda_\ell)^{-1})$; for $1 \leq j \leq n < \ell \leq m + n$, $\alpha_{j\ell}$ is $O(\lambda_j^{-1})$, and for $1 \leq j, \ell \leq m$, $\alpha_{j+n\,\ell+n} = b^{j\ell}(1 + O(\rho))$. The constants for the O-terms are bounded in terms of $m + n$, $\det B^{-1}$ and $\max\{|a_{j\ell}|, |b_{j\ell}|\}$.

By way of interest we recall the normal form for the metric dg_{WP}^2; the result is an immediate consequence of the above Theorem [Wol03, Cor. 4].

Corollary 8. *For the prescribed hyperbolic metric plumbing coordinates:*

$$dg_{WP}^2(s, t) = \left(dg_{WP}^2(s, 0) + \pi^3 \sum_{k=1}^{n} (4 dr_k^2 + r_k^6 d\theta_k^2) \right) (1 + O(\|r\|^3))$$

for $r_k = (-\log|t_k|)^{-1/2}$, $\theta_k = \arg t_k$ and $r = (r_1, \ldots, r_n)$.

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